High-frequency limit of the inverse scattering problem — from inverse Helmholtz to inverse Liouville

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Two inverse scattering problems

Objective: We propose a generalized inverse Helmholtz scattering problem and consider its connection to the inverse Liouville scattering problem in the high-frequency limit.

- **Inverse Helmholtz scattering problem:** We consider the Helmholtz equation with a source term $S^k(x)$
$$\Delta u^k + k^2 n(x) u^k = S^k(x)\,, \quad x \in \Omega \subset \mathbb{R}^d\,, \tag{1}$$
where u^k is the wave-field, k is the wave-number and $n(x)$ is the unknown medium. We aim to reconstruct $n(x)$ by probing the medium with different $S^k(x)$ and measuring the near-/far-field data.
- **Inverse Liouville scattering problem:** We consider the Liouville equation with a delta source in the phase space
$$v \cdot \nabla_x f + \frac{1}{2}\nabla_x n \cdot \nabla_v f = \delta(x - x_s)\delta(v - v_s)\,, \quad x \in \Omega, v \in \mathbb{S}^{d-1}\,, \tag{2}$$
where f is the distribution of photon particles. We aim to reconstruct $n(x)$ by injecting particles at different location x_s and velocity v_s, and then measuring the outgoing data.

▶ It is well-known that the Helmholtz equation converges to the Liouville equation in the **high frequency limit** ($k \to \infty$) by taking the **Wigner transform** $W^k[u^k]$.

▶ The above two inverse problems suggest different stability properties. The traditional inverse scattering problem is **ill-posed**, while the inverse Liouville equation is **well-conditioned**.

Convergence from the Helmholtz to the Liouville

The generalized inverse Helmholtz problem can be linked to the inverse Liouville problem by evaluating the convergence of the measurements.

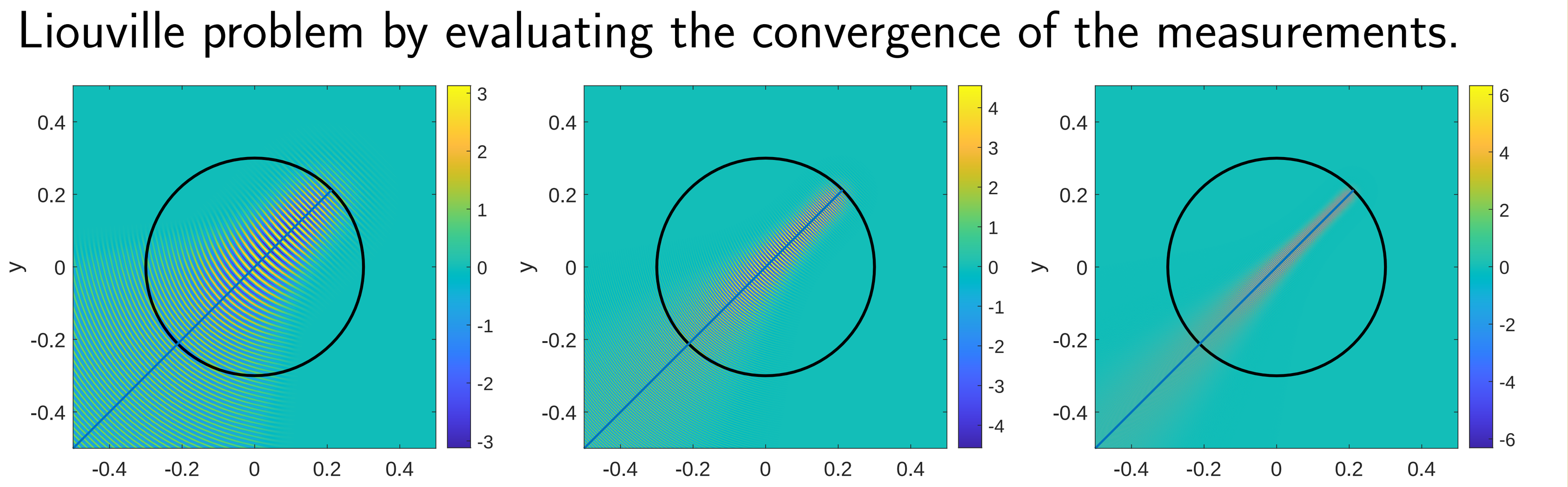

Figure 1: The real part of u^k for $k = 2^9$ (left), 2^{10} (middle) and 2^{11} (right). The blue lines show the Liouville trajectories.

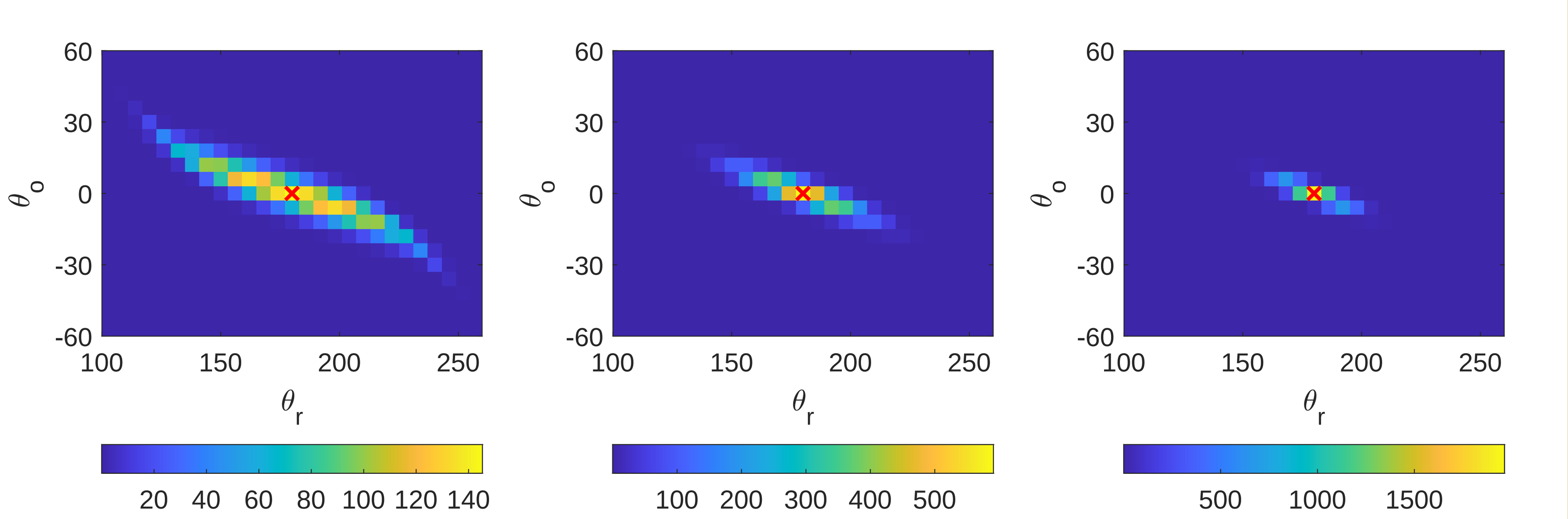

Figure 2: The Husimi transform $H^k u^k$ for $k = 2^9$ (left), 2^{10} (middle) and 2^{11} (right). θ_r denotes the receiver position and θ_o denotes the receiver direction. The red crosses denotes the Liouville data.

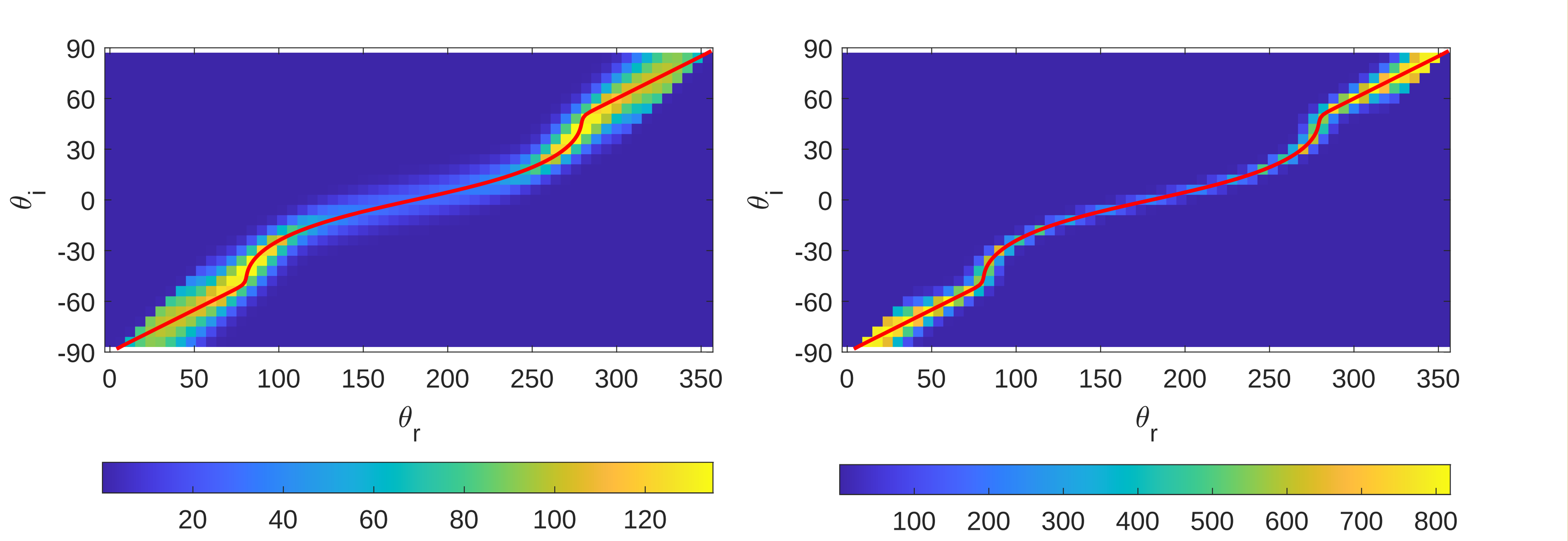

Figure 3: The angular-averaged Husimi transform $M_o^k(x)$ for $k = 2^9$ (left) and 2^{11} (right). θ_i denotes the incident direction. The red lines denotes the Liouville data.

The generalized inverse Helmholtz scattering problem

When the source and measurement are accordingly adjusted, the new formulation, called the **generalized inverse Helmholtz scattering**, are equivalent to the Liouville problem in the $k \to \infty$ limit:

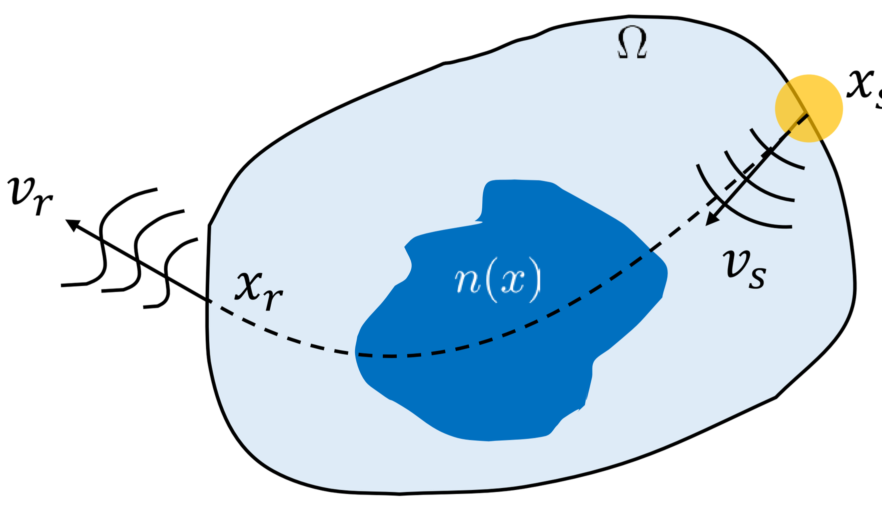

Figure 0: An illustration of the inverse scattering problem setup.The outgoing/incoming boundary $\Gamma_\pm = \{(x, v) : x \in \partial\Omega, , \pm\nu \cdot x > 0\}$

▶ Tightly concentrated monochromatic beams are impinged as source
$$S^k(x) \propto \chi(\sqrt{k}(x - x_s)) \exp\left(\mathrm{i}k v_s \cdot (x - x_s)\right)\,, \quad x \in \mathbb{R}^d\,, \tag{3}$$
where $\chi(x)$ is a bump function concentrated near the original.

▶ The wave-field is measured through the **Husimi transform**
$$H^k u^k(x_r, v_r) \propto \left|u^k * \phi^k_{v_r}(x_r)\right|^2\,, \quad (x_r, v_r) \in \Gamma_+\,, \tag{4}$$
where $\phi^k_v(x) = \chi(\sqrt{k}x)\exp(-\mathrm{i}kv \cdot x)$

▶ **Stable reconstruction** can be achieved in the high-frequency regime

- **Theorem 1:** As $k \to \infty$, the Wigner transform $W^k[u^k] \to f$.
- **Theorem 2:** As $k \to \infty$, the measurement data $H^k u^k \to f$.

Inversion Performance

The new inverse scattering formulation coupled with PDE-constrained optimization seems to be empirically less prone to cycle-skipping.

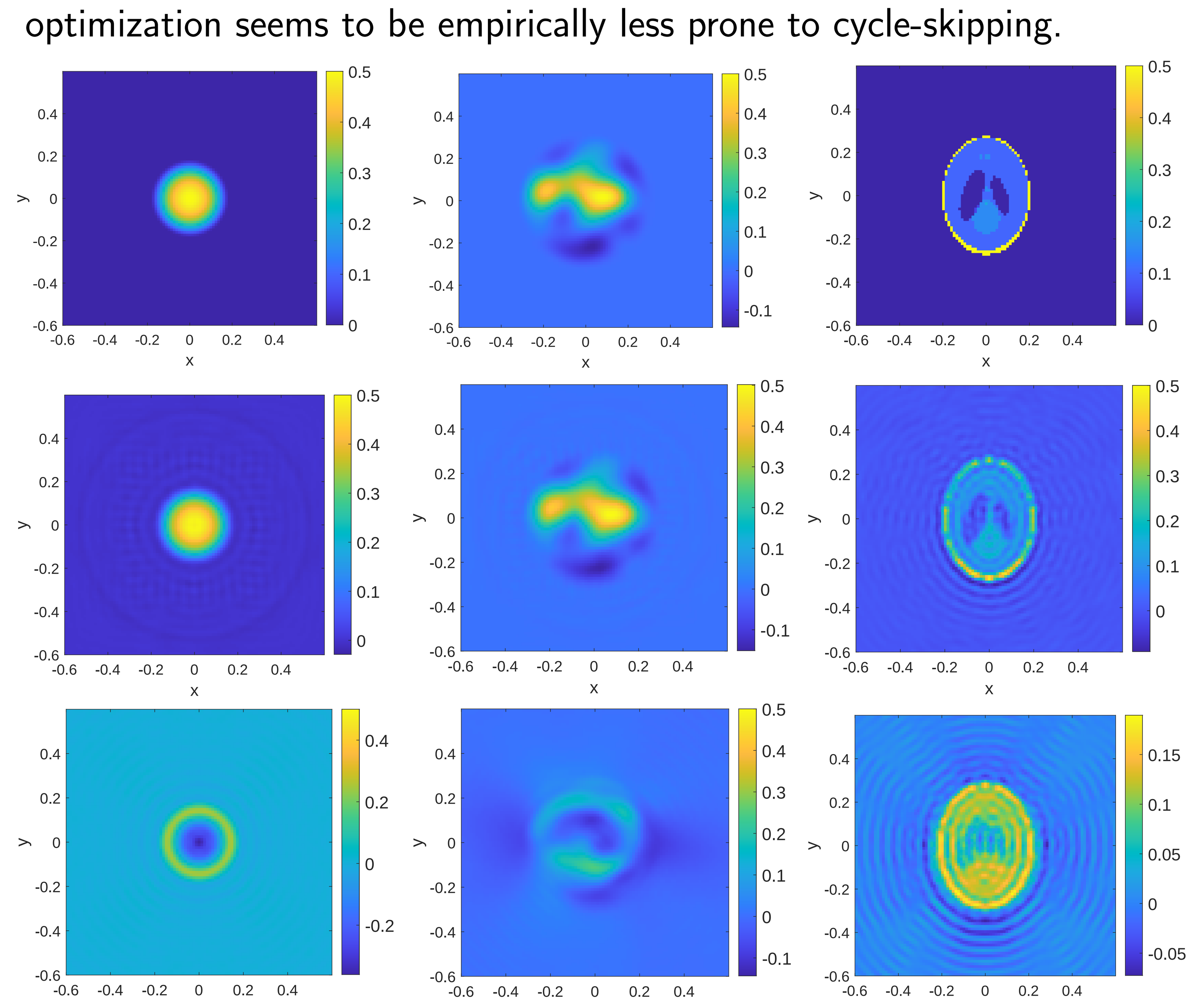

Figure 4: Three reconstruction examples. **First Row:** The ground truth media $n(x) - 1$: a bump function (left), a delocalized function (middle) and the Shepp-Logan phantom (right). **Second Row:** The reconstructed media by our new formulation. **Third Row:** The reconstructed media by the inverse Helmholtz scattering formulation with chromatic plane wave.

References

[1] S. Chen, Z. Ding, Q. Li, and L. Zepeda-Núñez, *High-frequency limit of the inverse scattering problem: asymptotic convergence from inverse helmholtz to inverse liouville*, To appear in SIAM Journal on Imaging Sciences. arXiv preprint arXiv:2201.03494, (2022).

arXiv:2201.03494, To appear in SIAM Journal on Imaging Sciences
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