

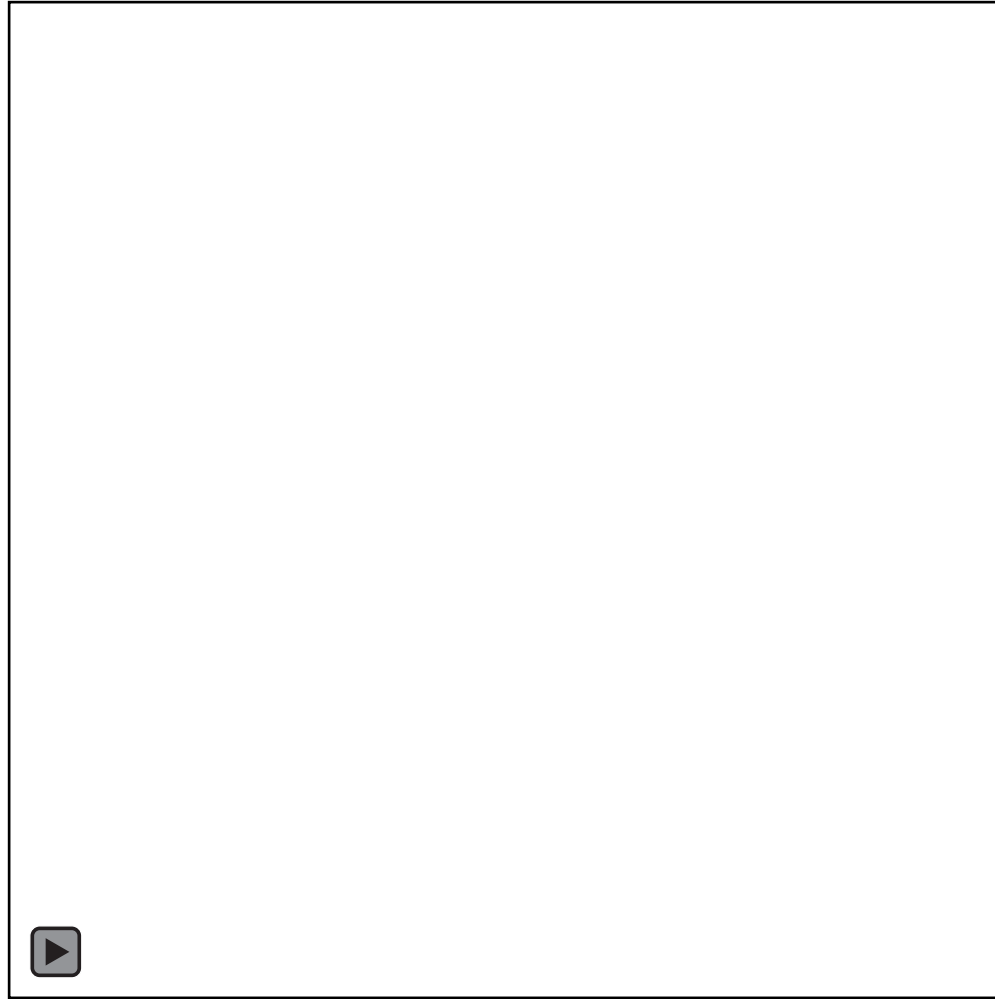
Knots and Colorability

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MIT 18.821 · Presentation Workshop · Fall 2026



Is it knotted — or just tangled?



Lower strands are
drawn with a gap

Is it knotted — or just tangled?



Lower strands are
drawn with a gap

GOAL OF THIS TALK

Decide whether a loop can be untangled without passing through itself

A harder one

Loop A

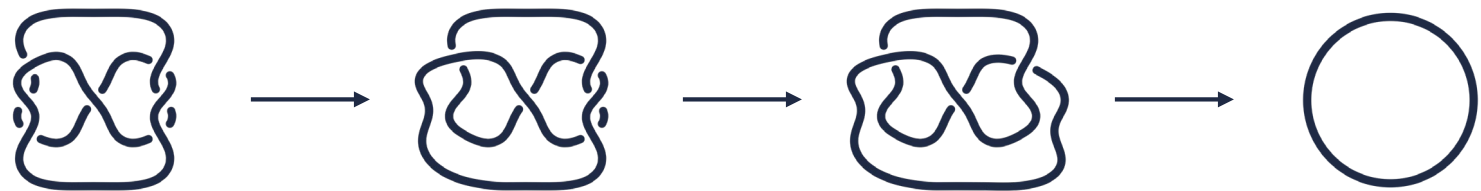


Same question: knotted, or just tangled?

Which one is knotted?



Loop A



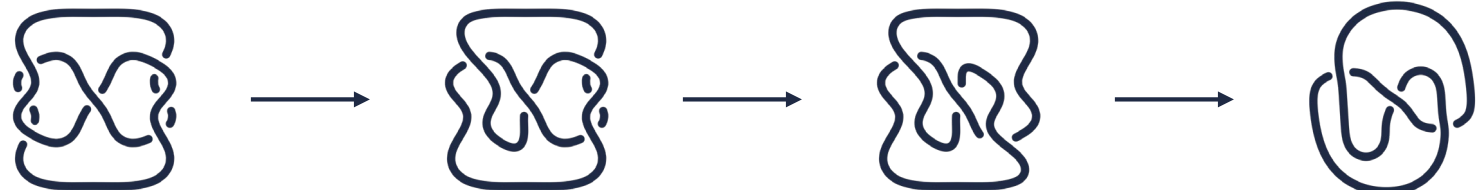
shift (left)

shift (right)

opens up: a circle

✓ a proof

Loop B



shift (left)

shift (right)

stuck at 3 crossings

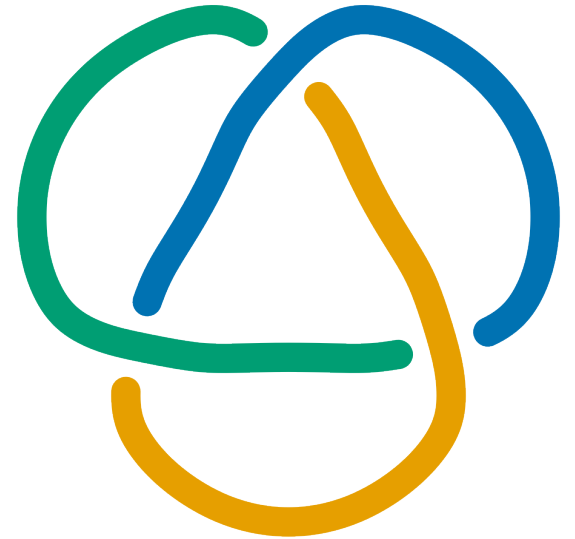
?

It is knotted
but how to prove

Use coloring

Outlines of the talk

- What is a knot? How to represent a knot?
- How can we tell whether two knots are the same?
- What is 3-colorability of knots?



Trefoil

Knot

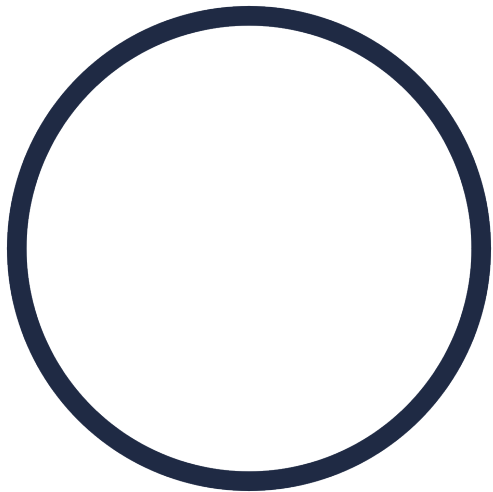
DEFINITION (KNOT)

Knot: a closed curve in $\mathbb{R}^3$ that does not intersect itself

DEFINITION (EQUIVALENCE)

Equivalence: K_0 and K_1 are equivalent if an ambient isotopy of $\mathbb{R}^3$ carries one onto the other

deform the rope continuously



the unknot

=



the unknot in disguise (loop A)

≠ ?

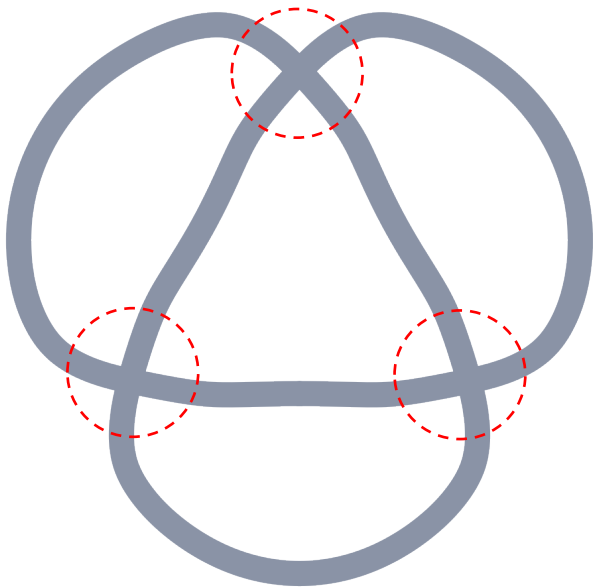


the trefoil (loop B)

Knot diagrams

DEFINITION (KNOT DIAGRAM)

Knot diagram: a projection of the knot onto a plane in which every crossing records which strand is on **top** or **bottom**

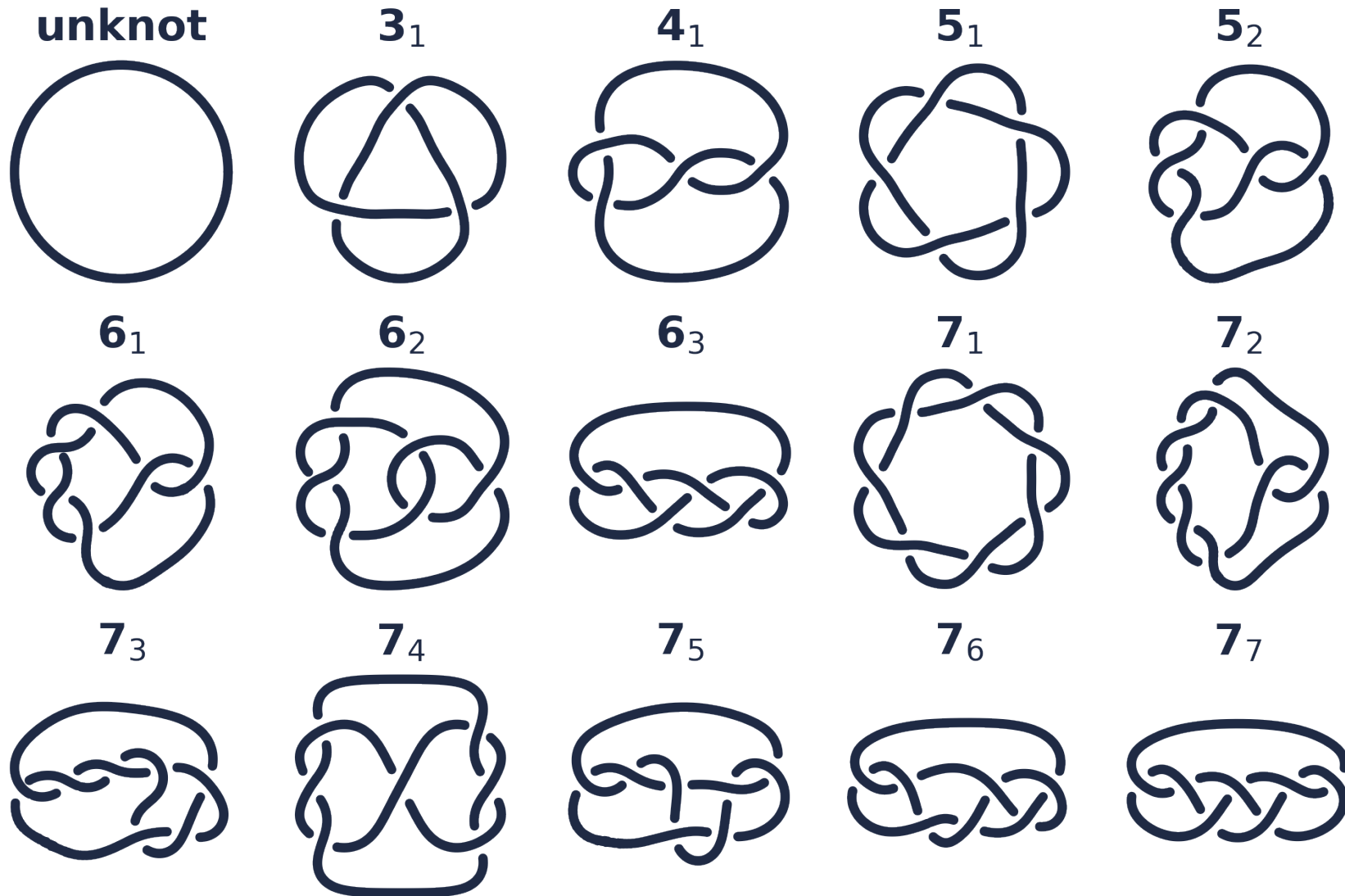


record over / under



the lower strand is drawn with a gap

A zoo of knots



Prime knot with at most 7 crossings

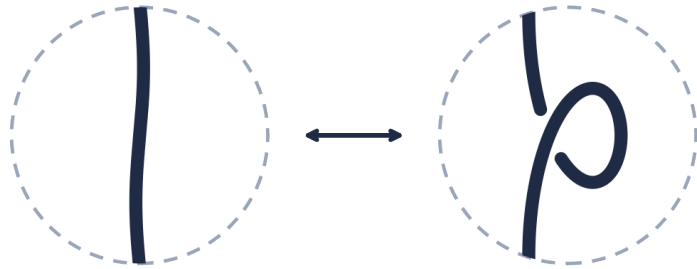
n: #crossings

k: catalogue index

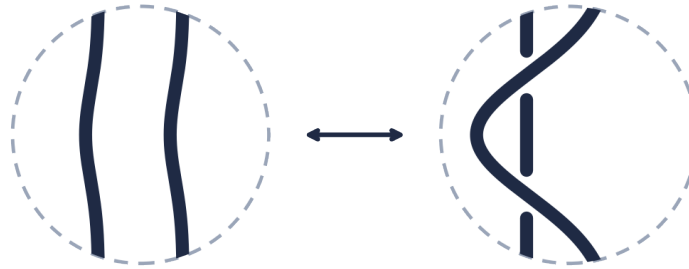
Are these all different?

Equivalence using diagram representation

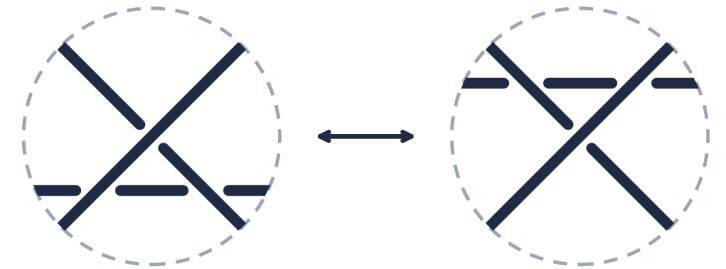
R-moves:



R-I twist



R-II poke



R-III slide

THEOREM (REIDEMEISTER, 1927)

Two diagrams represent equivalent knots **if and only if** one can be turned into the other by a finite sequence of the **three** moves below.

To **prove** equivalence: find a finite sequence of R-moves

To **disprove** equivalence?

Invariants

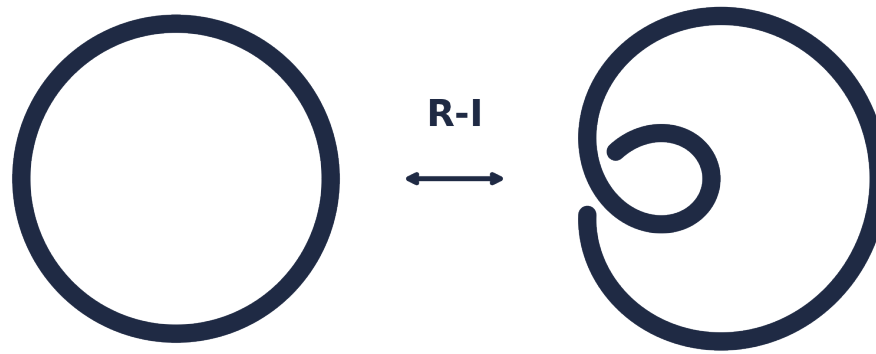
Disproof strategy: find a discriminator, computable from a drawing, that the R-moves cannot change.

DEFINITION (INVARIANT)

Knot invariant: a property of diagrams that is unchanged by the three R-moves.

THE STRATEGY

If $I(D_1) \neq I(D_2)$ then D_1 and D_2 show different knots.



Non-example: the number of crossings

Three-colorability

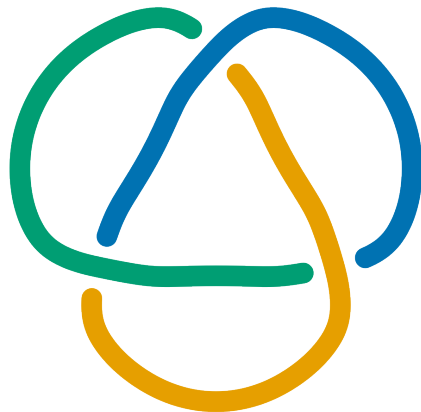
DEFINITION (3-COLORING)

Arc: a piece of the diagram drawn without a break (from one under-crossing to the next).

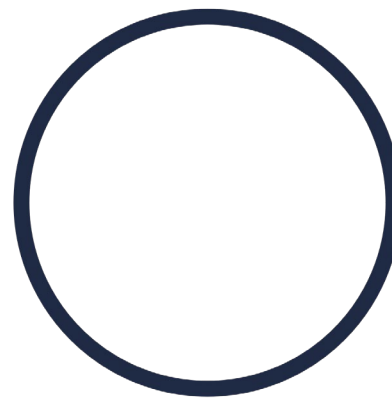
3-coloring: assigns one of three colors to every arc so that

1. at every crossing the three arcs that meet are all the same color or all different, and
2. at least two colors are used.

A diagram is **3-colorable** if it has a 3-coloring.



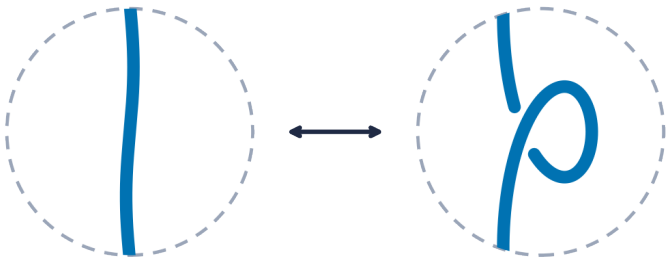
Trefoil: Colorable.



Circle: Not colorable.

Theorem: 3-colorability is a knot invariant

Proof: Each move changes the drawing **inside** a disk; the colors on the **boundary** of the disk are unchanged



R-I

The new crossing sees one color three times — all the same. ✓



R-II

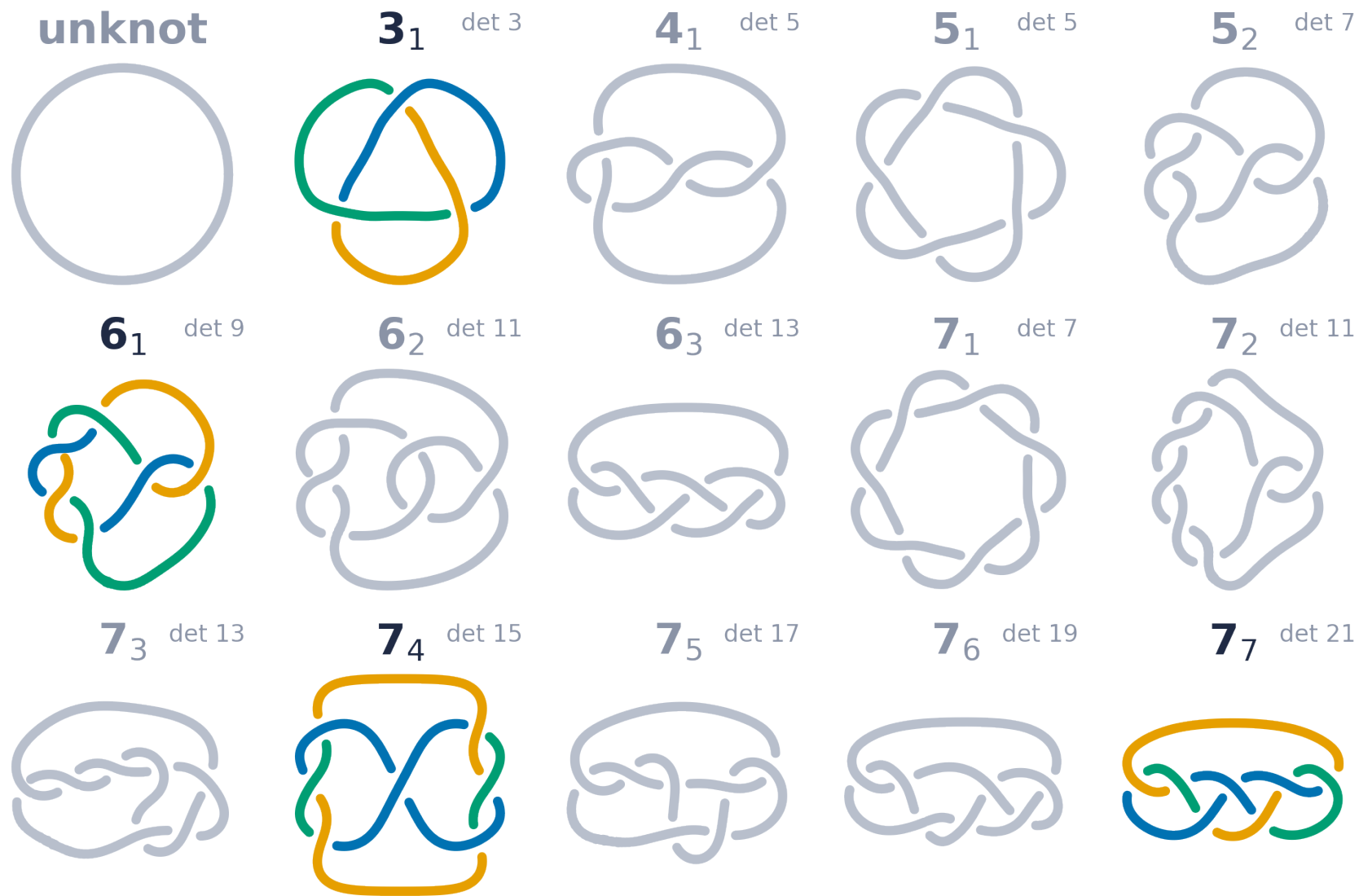
The trapped piece takes the third color; both new crossings then see all three colors. ✓



R-III

The inside can be recolored in exactly one way. ✓

A zoo of knots



More powerful tools than 3-coloring?

3-coloring is weird, but elegant (and useful)!



Crossing equations

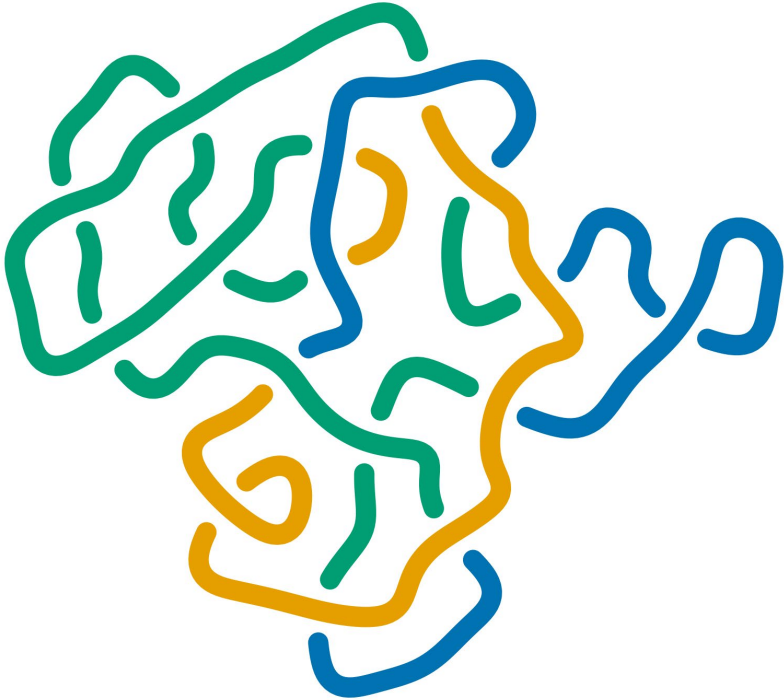
Colors 0, 1, 2. At every crossing:

$$2 \cdot (\text{over}) \equiv (\text{under}_1) + (\text{under}_2) \pmod{3}$$

LINEAR ALGEBRA MOD 3

- One equation per crossing, one unknown per arc.
- Non-trivial solution gives the explicit coloring.
- A non-trivial solution exists $\Leftrightarrow$ 3 divides the **determinant** of the knot

3-coloring is weird, but elegant (and useful)!



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Three things to take home

1

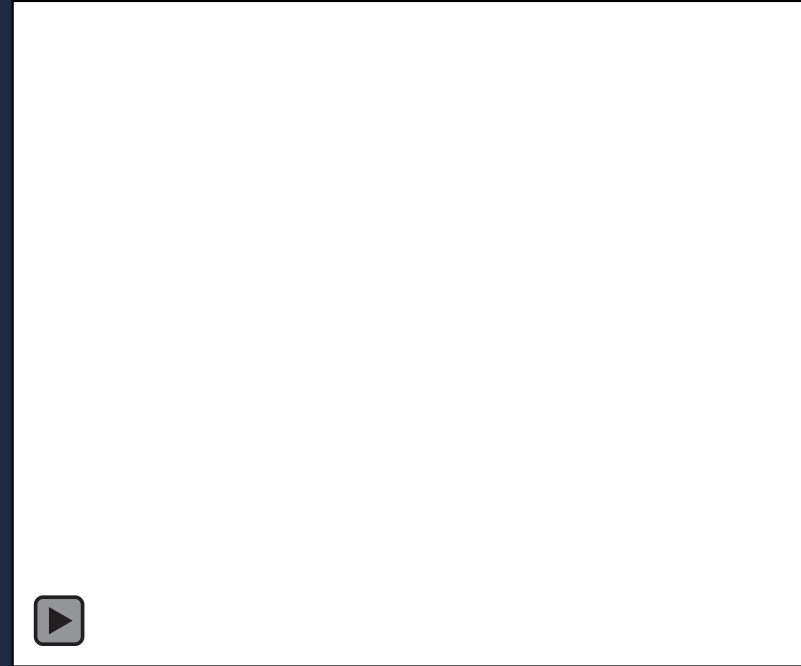
Knots are loops in space up to deformation

2

To prove two knots are different, find an invariant that differs.

3

3-colorability proves that the trefoil is knotted.



Thank you